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ON THE FAST ROTATION OF A HEAVY RIGID BODY ABOUT A FIXED POINT*

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Fast rotation of a symmetric heavy rigid body about a fixed point (the kinetic energy is large in comparison with the potential) is considered in cases when the resonance equations of Euler's motion /1,2/ are approximately satisfied at the initial instant (the body is assumed to effect $\sim 1/\varepsilon$ turns, ε is small, during time $t \sim 1$. It is shown that during that time $(t \sim 1)$ a finite deviation from inertial motion takes place. Such mechanical effect is similar to the precession of a fast top, except that it is more "early" (in the considered time scale the top precession is slow). Approximate equations that define the motion in the principal order and are integrable in quadratures. The formal process of derivation of higher approximations is indicated, and a geometric interpretation of motions is given.

The concept of motion of a fast rotation of a heavy rigid body presented in /1/ assumes that the momentum vector "M performs a slow precession about the vertical, and the body rotation relative to M approximately conforms to Poinsot's equations". This is inaccurate: the two approximate integrals $M^2 = A^2(p^2+q^2) + C^2r^2 \approx \text{const}$ and $2T = A(p^2+q^2) + Cr^2 \approx \text{const}$ may degenerate into one, for example, when r is small they degenerate into the integral $p^2 + q^2 \approx \text{const}$) and the approximate Poinsot motion does not generally follow from them, even during the time in which the precession of M can be neglected. Here we consider cases when a finite divergence from Poinsot's motion actually occurs.

1. The equations of motion and the estimate of accuracy. Let us consider the "fast" rotation introducing the small parameter ε so that during the time $t \sim 1$ the body performs $\sim 1/\varepsilon$ turns. We denote by $p/\varepsilon, q/\varepsilon, r/\varepsilon$ the projections of the absolute angular velocity of the body on the principal axes of inertia relative to a fixed point. For a symmetric body the equations of motion can then be written as (see, e.g., /3/)

$$\varepsilon\frac{dp}{dt} = \alpha qr - \varepsilon^2\zeta\gamma', \quad \varepsilon\frac{dq}{dt} = -\alpha pr + \varepsilon^2[\zeta\gamma - (1-\alpha)\xi\gamma''] \tag{1.1}$$
$$\varepsilon\frac{dr}{dt} = \varepsilon^2\xi\gamma', \quad \varepsilon\frac{d\gamma}{dt} = r\gamma' - q\gamma'', \quad \varepsilon\frac{d\gamma'}{dt} = p\gamma'' - r\gamma, \quad \varepsilon\frac{d\gamma''}{dt} = q\gamma - p\gamma'$$

where $\gamma, \gamma', \gamma''$ are the directional cosines of the vertical relative to these axes, the constant α is related to the principal moments of inertia A, B, C by the formula $A = B = C/(1-\alpha)$, and the constants ξ and ζ are proportional to deviations of the center of mass from the axis of symmetry and the equatorial plane of the ellipsoid of inertia, respectively.

The initial conditions for (1.1) must be subjected to the condition $p^2 + q^2 + r^2 \sim 1$ (it will obviously remain valid by virtue of the energy integral).

We shall call problem (1.1) perturbed, and problem

$$\varepsilon\frac{dp}{dt} = \alpha qr, \quad \varepsilon\frac{dq}{dt} = -\alpha pr, \quad \varepsilon\frac{dr}{dt} = 0; \quad \varepsilon\frac{d\gamma}{dt} = r\gamma' - q\gamma'', \quad \varepsilon\frac{d\gamma'}{dt} = p\gamma'' - r\gamma, \quad \varepsilon\frac{d\gamma''}{dt} = q\gamma - p\gamma' \tag{1.2}$$

(with the same initial data) unperturbed. Thus the unperturbed problem is defined by the Euler equations of motion by inertia.

We shall consider problem (1.1) guided by the concept of the theory of perturbations, which is based on the attempt of choosing a change of variables close to identical, i.e. to set

$$p_* = p + \varepsilon f_1(p, \ldots, \gamma, \ldots) + \ldots + \varepsilon^k f_k(p, \ldots; \gamma, \ldots), \ldots) \tag{1.3}$$

so as to have Eqs. (1.1) in the new variables $p_*, \ldots, \gamma_*, \ldots$ of a simpler form, but accurate within $\sim\varepsilon^{k+1}$.

The estimate of accuracy is based here on the following simple fact.

Theorem. Let us assume that the solution $\varphi_\nu(t, \varepsilon), \lambda_\mu(t, \varepsilon)$ of the perturbed system of special form

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$$\varepsilon\frac{d\varphi_\nu}{dt}=i\lambda_\nu\varphi_\nu+\varepsilon^m F_\nu(\lambda,\varphi,\varepsilon),\quad \varepsilon\frac{d\lambda_\mu}{dt}=\varepsilon^m G_\mu(\lambda,\varphi,\varepsilon),\qquad \nu=1,\ldots,l,\ i^2=-1,\ \mu=1,\ldots,l,\ldots,n \tag{1.4}$$

satisfies for certain initial data the following conditions: 1) λ_ν are real, and 2) $\varphi_\nu(t,\varepsilon)$, $F_\nu(\lambda(t,\varepsilon),\ \varphi(t,\varepsilon),\ \varepsilon)=f_\nu(t,\varepsilon)$, $G_\mu(\lambda(t,\varepsilon)$, and $\varphi(t,\varepsilon),\varepsilon)=g_\mu(t,\varepsilon)$ are for $0\leqslant t\leqslant C(\varepsilon)=O(1)$ (i.e. $t\leqslant 1$) bounded by constants independent of ε and integrable with respect to t.

Let, moreover, $\Phi_\nu(t,\varepsilon)$, $\Lambda_\mu=\text{const}$ be the solution of the unperturbed system

$$\varepsilon\frac{d\Phi_\nu}{dt}=i\Lambda_\nu\Phi_\nu,\quad \varepsilon\frac{d\Lambda_\mu}{dt}=0 \tag{1.5}$$

for the same initial data as for φ_ν and λ_μ.

Then, as $\varepsilon\to 0$

$$|\lambda_\mu-\Lambda_\mu|=O(\varepsilon^{m-1}),\quad |\varphi_\nu-\Phi_\nu|=O(\varepsilon^{m-2})$$

The proof follows from formulas

$$\Lambda_\mu=\lambda_\mu(t,\varepsilon)-\varepsilon^{m-1}\int_0^t g_\mu(\tau,\varepsilon)\,d\tau,\quad \Phi_\nu(t,\varepsilon)=\varphi_\nu(t,\varepsilon)-$$
$$\varepsilon^{m-2}\exp\left(\frac{i\Lambda_\nu t}{\varepsilon}\right)\int_0^t\left[i\varphi_\nu(\tau,\varepsilon)\int_0^\tau g_\nu(\xi,\varepsilon)\,d\xi+\varepsilon f_\nu(\tau,\varepsilon)\right]\times\exp\left(-\frac{i\Lambda_\nu\tau}{\varepsilon}\right)d\tau \tag{1.6}$$

which can be tested by substitution into (1.5).

Remark 1.1. For large $t\leqslant 1/\varepsilon^k$ the estimate can be similarly obtained. In our investigation we restrict the time to $0\leqslant t\leqslant 1$.

Formulas (1.6) show that the over-all approximation error $\varphi_\nu=\Phi_\nu,\lambda_\mu=\Lambda_\mu$ is a quantity of order ε^{m-2} which is generally small only for $m>2$. When $G_\nu=O(\varepsilon)$, the error is obviously small also for $m=2$. Note that when $m=2$ the first integrals of the unperturbed system are approximate integrals of the perturbed system.

Turning to Eqs. (1.1) and (1.2) we set

$$\varkappa=p\gamma+q\gamma'+(1-\alpha)r\gamma'',\quad \varphi_{1,2}=q\pm ip$$
$$\varphi_{3,4}=-k^2\gamma''+(1-\alpha)r\varkappa\pm ik(q\gamma-p\gamma'),\quad k^2=p^2+q^2+(1-\alpha)^2r^2 \tag{1.7}$$

and under the condition that

$$\omega^2=p^2+q^2\sim 1 \tag{1.8}$$

where $\omega=0$ is a singular point for the substitution (1.7), pass to the new variables $\alpha r,\varkappa$, $\varphi_1,\varphi_2,\varphi_3,\varphi_4$. Euler's problem (1.2) assumes the form

$$\varepsilon\frac{d\alpha r}{dt}=0,\quad \varepsilon\frac{d\varkappa}{dt}=0,\quad \varepsilon\frac{d\varphi_1}{dt}=i\alpha r\varphi_1,\quad \varepsilon\frac{d\varphi_2}{dt}=-i\alpha r\varphi_2,\quad \varepsilon\frac{d\varphi_3}{dt}=ik\varphi_3,\quad \varepsilon\frac{d\varphi_4}{dt}=-ik\varphi_4,\quad \varepsilon\frac{dk}{dt}=0 \tag{1.9}$$

where k is a supplementary ancillary variable, i.e. the form (1.5), and the perturbed system (1.1) assumes, respectively, the form (1.4), and $m=2$.

Further consideration of (1.8) with allowance for the above estimate enables us to conclude that: 1) closeness to the initially Euler's motion (i.e. during time $t\leqslant\varepsilon$) cannot generally be guaranteed during the time $t\leqslant 1$, and 2) if the substitution (1.3) can be such that in the new variables $m\geqslant 3$ or, at least in the equations for $\lambda_{1,2}=\pm\alpha r$ and $\lambda_{3,4}=\pm k$ the perturbation is $\leqslant\varepsilon^3$, the motion remains close to Euler's motion by inertia for $t\leqslant 1$. Condition (1.8) remains satisfied, if it was satisfied at the initial instant, since $\omega=\text{const}$ is the integral of Euler's motion (we recall that by virtue of (1.2) $A=B$ and $d\omega/dt=0$.

Let us first clarify when such substitution exists and, then concentrate our attention on cases when it is impossible (for the case of small ω^2 see Remark 5.2).

We begin by expounding the formalism of the theory of perturbations which will be subsequently used.

2. The formalism of the theory of perturbations.

We shall use a modification of the formalism in /4/ which is based on Lie transformation, which differs from the Deprit–Hori procedure in the application of Hausdorf's formula, which in the authors' opinion simplifies calculations. We present a purely formal exposition, bearing in mind that all expansions in the small parameter are assumed to be brought to a fixed order, and that the closeness of solution of the obtained approximate system of equations to the exact solution is to be established after its investigation.

Let us consider the system of differential equations

$$\varepsilon \frac{dx_j}{dt} = f_{j0}(x) + \varepsilon f_{j1}(x) + \ldots \quad (j = 1, \ldots, n; \quad x = \{x_1, \ldots, x_n\}) \tag{2.1}$$

and the equivalent to it first order equation in partial derivatives

$$\varepsilon \frac{\partial F(t, x)}{\partial t} = LF \equiv (L_0 + \varepsilon L_1 + \varepsilon^2 L_2 + \ldots) F(t, x), \qquad L_i = \sum_{j=1}^{n} f_{ji}(x) \frac{\partial}{\partial x_j} \quad (i = 0, 1, \ldots) \tag{2.2}$$

where L_i are linear differential operators of the first order (so that system (2.1) is characteristic for L).

All functions considered here are assumed bounded and fairly smooth in some domain D of variation of variables x . We introduce new variables x_* using the formula

$$x_* = e^M x, \quad M = \sum_{j=1}^{n} \mu_j(x, \varepsilon) \frac{\partial}{\partial x_j} \tag{2.3}$$

where M is some first order operator. It can be shown that in the new variables Eq.(2.2) is of the form /4/

$$\varepsilon \frac{\partial F(t, x)}{\partial t} = XF(t, x) \equiv e^{-M} L e^M F(t, x) \tag{2.4}$$

where, and in what follows the asterisk at x is omitted for brevity. It should be pointed out in this connection that for reverting to the old variables it is necessary to apply the inverse transform e^{-M}.

Let the substitution of variables be close to the identical

$$M = \varepsilon M_1 + \varepsilon^2 M_2 + \ldots \tag{2.5}$$

where M_i is independent of ε.

Using the Hausdorf formula

$$e^{-M} L e^M = L + [LM] + \frac{1}{2!}[[LM]M] + \ldots$$

where $[AB] = AB - BA$ is the commutator, we obtain Eq. (2.4) of a form similar to (2.2)

$$\varepsilon \frac{\partial F(t, x)}{\partial t} = XF(t, x) = (X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \ldots) F(t, x) \tag{2.6}$$

$$X_0 = L_0, \quad X_1 = [L_0 M_1] + L_1, \quad X_2 = [L_0 M_2] + [L_1 M_1] + {}^1/_2 [[L_0 M_1] M_1] + L_2$$

Formulas for X_i are of the recurrent type

$$X_i = [L_0 M_i] + N_i \tag{2.7}$$

where N_i is a known operator when $M_1, \ldots, M_{i-1}$ are specified.

We can attempt to simplify Eq.(2.6) in comparison with (2.2) by the selection of (2.5), i.e. of M_i.

The question arises of the possibility of simpler properties of X_i.

The answer is suggested by the following algebraic analogy. Let L and M be square matrices. If L_0 is a diagonal (or reducible to a diagonal) matrix, a simple reasoning shows that it is possible by the selection of M_i (for an arbitrary a priori specified Matrix N_i) to obtain the property $[L_0 X_i] = 0$. Generally (when L_0 is a Jordan or reducible to a Jordan matrix) it is possible to obtain $[L_0[L_0[L_0 \ldots [L_0 X_i]]]] = 0$, where the degree of commutation depends on the largest dimension of the Jordan lattice in L_0.

We introduce operator l_0 acting on the first order operators (N) in conformity with formula $l_0 N = [L_0 N]$ and, guided by the explained analogy, assume that by definition the simplest property which it is possible to specify for X_i is

$$l_0^{\nu} X_i \equiv [L_0[L_0 \ldots [L_0 X_i]]] = 0 \tag{2.8}$$

We shall indicate here only the sufficient condition for satisfying (2.8). Let operator N_i be such that

$$\omega_\nu(x)\, l_0^{\nu} N_i + \omega_{\nu+1}(x)\, l_0^{\nu+1} N_i + \ldots + \omega_k(x)\, l_0^{k} N_i = 0 \tag{2.9}$$

where $\nu \geqslant 0$ and functions $\omega_s(x)$ $(s = \nu, \ldots, k)$ are the invariants of operator L_0: $L_0 \omega_s = 0$ and $\omega_\nu(x)$ and do not vanish at any point of the domain D. It is then possible to satisfy (2.8)

$$M_i = \sum_{j=1}^{k-\nu} \frac{\omega_{\nu+j}}{\omega_\nu} l_0^{j-1} N_i, \quad X_i = \sum_{j=0}^{k-\nu} \frac{\omega_{\nu+j}}{\omega_\nu} l_0^j N_i \tag{2.10}$$

When $\nu = 0$ we obtain $X_i = 0$. We shall call equality (2.9) the commutative relation for N_i.

Using the Jacobi identity $l_0[AB] = [l_0A, B] + [A, l_0B]$ we can prove the following simple theorem. If each of the operators L_i satisfies some commutative relation with constant coefficients, a commutative relations can be also found for each of the operators

$$N_1 = L_1, \quad N_2 = [L_1M_1] + {}^1/_2[[L_0M_1]\, M_1] + L_2, \ldots$$

appearing in (2.7), and, consequently, problem (2.8) is solvable in any order.

3. **The case of arbitrary intitial conditions.** For system (1.1) we have

$$L_0 = \alpha r \left(q \frac{\partial}{\partial p} - p \frac{\partial}{\partial q}\right) + (r\gamma' - q\gamma'') \frac{\partial}{\partial \gamma} + (p\gamma'' - r\gamma) \frac{\partial}{\partial \gamma'} + (q\gamma - p\gamma') \frac{\partial}{\partial \gamma''} \tag{3.1}$$

$$L_1 = 0, \quad L_2 = \xi \left\{-(1-\alpha)\gamma'' \frac{\partial}{\partial q} + \gamma' \frac{\partial}{\partial r}\right\} + \zeta\left(-\gamma' \frac{\partial}{\partial p} + \gamma \frac{\partial}{\partial q}\right)$$

Setting $M_1 = 0$ we obtain $X_1 = 0$, $X_2 = [L_0M_2] + L_2$.

If an operator M_2 is such that $X_2 = 0$ can be found, system (1.1) is reduced to Euler's system with an accuracy $\sim \varepsilon^3$ and in conformity with the estimate obtained in Sect.1 the system remains close to Eulerian during the time $t \leqslant 1$. We would recall that for this it is sufficient only that $\varepsilon d\lambda_\nu / dt = O(\varepsilon^3)$. Consequently, the motion remains close to unperturbed one, if it is possible to obtain $X_2 r = 0$, $X_2 k = 0$ (since in the considered problem $\lambda_{1,2} = \pm \alpha r$, $\lambda_{3,4} = \pm k$, see (1.9)).

By virtue of (3.1) we have

$$X_2 r = L_0 M_2 r + \xi\gamma', \quad X_2 \frac{k^2}{2} = L_0 M_2 \frac{k^2}{2} + L_0 \{(1-\alpha)\xi\gamma + \zeta\gamma''\} - \alpha(1-\alpha)\xi r\gamma'$$

If the quantity γ' can be represented in the form $L_0 f$, it is possible to obtain the equality $X_2 r = 0$, $X_2 k = 0$ by selecting $M_2 r$, $M_2 k^2/2$, and it is easily to calculate

$$L_0^2 \gamma' = -2\alpha r L_0 \gamma - (k^2 - \alpha^2 r^2)\gamma' + \frac{\varkappa}{\alpha r} L_0 p$$

which shows that for $\alpha r \neq 0$, $k^2 - \alpha^2 r^2 \neq 0$ the quantity γ' can be represented in the form $L_0 f$.

Excluding the case of total kinetic symmetry $\alpha = 0$ we conclude that during the time $t \leqslant 1$ motion of the body remains close to the initial Eulerian motion for all initial data ($\omega \neq 0$), possibly except two cases, viz: 1) when at the initial instant r is small, and 2) when $k^2 - \alpha^2 r^2$ is small at $t = 0$.

These are cases of resonance in Euler's motion (1.9) (see, e.g., /1,2/).

4. **The first case.** Let r be small at the initial instant (so that $\omega^2 = p^2 + q^2 \sim 1$). Substituting εr for r in (1.1) we obtain the system

$$\varepsilon \frac{dp}{dt} = \varepsilon \alpha q r - \varepsilon^2 \zeta \gamma', \quad \varepsilon \frac{dq}{dt} = -\varepsilon \alpha r p + \varepsilon^2 [\zeta\gamma - (1-\alpha)\xi\gamma''] \tag{4.1}$$

$$\varepsilon \frac{dr}{dt} = \varepsilon \xi \gamma' \quad \varepsilon \frac{d\gamma}{dt} = -q\gamma'' + \varepsilon r \gamma', \quad \varepsilon \frac{d\gamma'}{dt} = p\gamma'' - \varepsilon r \gamma, \quad \varepsilon \frac{d\gamma''}{dt} = q\gamma - p\gamma'$$

Calculations are conveniently carried out in the new variables

$$p,\ q,\ r,\ \gamma'',\ u = p\gamma + q\gamma',\ v = q\gamma - p\gamma' \tag{4.2}$$

in which L_0 is of the form $L_0 = v\partial/\partial\gamma'' - \omega^2\gamma''\partial/\partial v$ and ($p, q, r, u, v^2 + \omega^2\gamma''^2$ are the five invariants of the unperturbed problem).

The operator L_1 satisfies the commutational relation

$$\omega^2 l_0 L_1 + l_0^3 L_1 = 0 \quad (L_0 \omega^2 = 0) \tag{4.3}$$

Setting $M_1 = \omega^{-2} l_0 L_1$ we obtain in conformity with (4.3) and (2.10) X_1 in variables (4.2)

$$X_1 = \alpha r \left(q \frac{\partial}{\partial p} - p \frac{\partial}{\partial q}\right) + \frac{\xi}{\omega^2} q u \frac{\partial}{\partial r}, \quad l_0 X_1 = 0 \tag{4.4}$$

We write, without so far determining X_2, the ordinary system corresponding to (2.6)

$$\varepsilon \frac{dp}{dt} = \varepsilon \alpha r q + O(\varepsilon^2), \quad \varepsilon \frac{dq}{dt} = -\varepsilon \alpha r p + O(\varepsilon^2), \quad \varepsilon \frac{dr}{dt} = \varepsilon \frac{\xi}{\omega^2} q u + O(\varepsilon^2)$$

$$\varepsilon \frac{du}{dt} = O(\varepsilon^2), \quad \varepsilon \frac{dv}{dt} = -\omega^2 \gamma'' + O(\varepsilon^2), \quad \varepsilon \frac{d\gamma''}{dt} = v + O(\varepsilon^2)$$

For the determination of p, q, r, u with an accuracy to $\sim\varepsilon$ during times $t \leqslant 1$ the terms $O(\varepsilon^2)$ can evidently be discarded in the equations of these variables. As regards v and γ'', we set $\varphi_{1,2} = v \pm i\omega\gamma''$ and obtain

$$\varepsilon\frac{d\varphi_1}{dt} = i\omega\varphi_1 + O(\varepsilon^2), \quad \varepsilon\frac{d\varphi_2}{dt} = -i\omega\varphi_2 + O(\varepsilon^2), \qquad \varepsilon\frac{d\omega}{dt} = \varepsilon^2 X_2\omega + O(\varepsilon^3)$$

A reasoning similar to that in Sect.1 leads to the conclusion that, when the equality $X_2\omega = 0$ can be obtained, the terms $O(\varepsilon^2)$ can be rejected at $t \leqslant 1$ in these equations yielding an error of order ε. Taking into account that $L_0\omega = L_1\omega = 0$ and $M_1\omega = 0$ we obtain

$$X_2\frac{\omega^2}{2} = L_0 M_2\frac{\omega^2}{2} + L_2\frac{\omega^2}{2}, \quad L_2\frac{\omega^2}{2} = L_0\{(1-\alpha)\xi\gamma + \zeta\gamma''\}$$

and setting $M_2\omega^2/2 = -(1-\alpha)\xi\gamma - \zeta\gamma''$ we have $X_2\omega = 0$.

Thus the motion of the body during times $t \leqslant 1$ is defined in the considered here case with an accuracy to $\sim\varepsilon$ by the system of equations

$$\frac{dp}{dt} = \alpha qr, \quad \frac{dq}{dt} = -\alpha rp, \quad \frac{dr}{dt} = \frac{\xi}{\omega^2}qu, \quad \frac{du}{dt} = 0, \quad \varepsilon\frac{dv}{dt} = -\omega^2\gamma'', \qquad \varepsilon\frac{d\gamma''}{dt} = v \tag{4.5}$$

Besides the obvious integrals

$$\omega^2 = p^2 + q^2 = \text{const}, \ u = \text{const}, \ v^2 + \omega^2\gamma''^2 = \text{const}$$

which are approximate corollaries of the three classic integrals of motion, system (4.5) has the new integral

$$r^2 - \frac{2\xi}{\alpha\omega^2}up = \text{const}$$

and is integrable in quadratures.

Note that in this approximation the center of mass deviation (ζ) from the equatorial plane does not affect the motion.

Remark 4.1. Using the method described in Sect.3 it is possible to obtain approximations of any order. This is conveniently done by considering operators $\omega^{-1}L_0$, $\omega^{-1}L_1$, $\omega^{-1}L_2$ instead of operators L_0, L_1, L_2 (introduction of the multiplier ω^{-1} is equivalent to a change of time). The commutational relations for $\omega^{-1}L_1, \omega^{-1}L_2$ have then constant coefficients. It is obviously necessary to bear in mind that after solving the approximate system (and a check of accuracy), it is necessary to revert to initial variables using the transform e^{-M} (with an accuracy of the order of $\sim\varepsilon$ this is evidently unnecessary).

5. **The second case.** Investigation of the case when $k^2 - \alpha^2 r^2$ is small is similar to that of the first case. Setting in Eqs.(1.1)

$$r = n\omega + \varepsilon s \tag{5.1}$$

we select the constant n so that $k^2 - \alpha^2 r^2 \sim \varepsilon \ (s \sim 1)$. Then

$$\alpha = \frac{n^2+1}{2n^2}, \quad A = B = \frac{2n^2}{n^2-1}C \quad \left(1 > \alpha > \frac{1}{2}\right) \tag{5.2}$$

For system (1.1) we then have

$$L_0 = n\omega\left\{\alpha\left(q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\right) + \left(\gamma'\frac{\partial}{\partial\gamma} - \gamma\frac{\partial}{\partial\gamma'}\right)\right\} + \gamma''\left(-q\frac{\partial}{\partial\gamma} + p\frac{\partial}{\partial\gamma'}\right) + (q\gamma - p\gamma')\frac{\partial}{\partial\gamma''}$$

$$L_1 = s\left\{\alpha\left(q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\right) + \left(\gamma'\frac{\partial}{\partial\gamma} - \gamma\frac{\partial}{\partial\gamma'}\right)\right\} + \left\{\frac{\xi}{\omega}[\omega\gamma' + n(1-\alpha)q\gamma''] - \frac{n\zeta}{\omega}(q\gamma - p\gamma')\right\}\frac{\partial}{\partial s}$$

$$L_2 = -(1-\alpha)\xi\gamma''\frac{\partial}{\partial q} - \zeta\left(\gamma'\frac{\partial}{\partial p} - \gamma\frac{\partial}{\partial q}\right)$$

where it is convenient to pass to variables

$$p, \ q, \ s, \ I = n(1-\alpha)qu + n\alpha pv - \omega q\gamma'', \ \varkappa = u + n(1-\alpha)\omega\gamma'', \ I_* = n(1-\alpha)pu - n\alpha qv - \omega p\gamma'' \tag{5.3}$$

where n and v are the same as in (4.2). Then

$$L_0 = \alpha n\omega\left(q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\right) \tag{5.4}$$

The commutational relation for L_1 is

$$4(n\alpha\omega)^4 l_0 L_1 + 5(n\alpha\omega)^2 l_0^3 L_1 + l_0^5 L_1 = 0$$

Selecting M_1 in conformity with (2.10) we obtain X_1 in variables (5.3)

$$X_1 = \alpha s\left(q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\right) + \frac{s}{n^2\alpha}\left(I\frac{\partial}{\partial I_*} - I_*\frac{\partial}{\partial I}\right) - \frac{\xi}{2\omega^2 n\alpha} I\frac{\partial}{\partial s} \tag{5.5}$$

As in the previous case when $t \lesssim 1$ perturbations $\sim\varepsilon^2$ can be neglected in the equations for $s, \varkappa, I, I_*$, i.e. the expressions for X_2 in these variables are of no interest. In conformity with (5.5) the approximate equations for these are

$$\frac{ds}{dt} = -\frac{\xi}{2\omega^2 n\alpha} I, \quad \frac{d\varkappa}{dt} = 0, \quad \frac{dI}{dt} = -\frac{s}{n^2\alpha} I_*, \quad \frac{dI_*}{dt} = \frac{s}{n^2\alpha} I \tag{5.6}$$

As for p and q we set $\varphi_1 = q + ip$, $\varphi_2 = q - ip$ and from (5.4) and (5.5) obtain

$$\varepsilon\frac{d\varphi_{1,2}}{dt} = \pm i\alpha(n\omega + \varepsilon s)\varphi_{1,2} + O(\varepsilon^2), \quad \varepsilon\frac{d\omega}{dt} = \varepsilon^2 X_2\omega + O(\varepsilon^3) \tag{5.7}$$

To calculate φ_1 and φ_2 with an accuracy to ε for $t \lesssim 1$ it is necessary to determine ω with an error $\lesssim \varepsilon^2$ (see (1.6)). Therefore the terms $O(\varepsilon^2)$ can be rejected in (5.7) only if it is possible to obtain $X_2\omega = 0$. But in the case under consideration (as distinct from the previous one)

$$X_2\omega = L_0 M_2\omega + L_0 f + \frac{1-\alpha}{2n^2\alpha^2\omega^2}\xi I$$

where the last term is an invariant of L_0. A suitable selection of $M_2\omega$ yields

$$X_2\omega = \frac{1-\alpha}{2n^2\alpha^2\omega^2}\xi I$$

In accordance with this (with required accuracy)

$$\frac{d\omega}{dt} = \varepsilon(1-\alpha)\frac{\xi}{2n^2\alpha^2\omega^2} I \tag{5.8}$$

The approximate equations for p and q are now obtained in the form

$$\varepsilon\frac{dp}{dt} = \alpha(n\Omega + \varepsilon s)q, \quad \varepsilon\frac{dq}{dt} = -\alpha(n\Omega + \varepsilon s)p \tag{5.9}$$

where parameter Ω is determined as the solution of Eq.(5.8) which by virtue of (5.6) is integrable.

Thus, if in the initial instant the angular velocity vector lies close to the cone $p^2 + q^2 - r^2/n^2 = 0$, where n is defined in (5.2), the motion of the body (accurate within ε) during time $t \lesssim 1$ is defined by the system of Eqs.(5.6) and (5.9).

As in previous cases, the system besides the obvious integrals

$$\omega^2 = p^2 + q^2 = \text{const}, \quad \varkappa = \text{const}, \quad I^2 + I_*^2 = \text{const}$$

(approximate corollaries of classic integrals) the system has a fourth new integral

$$s^2 + \frac{\xi n}{\omega^2} I_* = \text{const}$$

and is integrable in quadratures.

Remark 5.1. Same as Remark 4.1.

Remark 5.2. The case when at the initial instant ω is small was not considered above. It corresponds to fast rotation about the axis of symmetry, and can be investigated in exactly the same way as the previous cases, by substituting in (1.1) εp and εq for p and q. During the time $t \lesssim 1$ the motion proves to remain close to the initial Eulerian, while the nontrivial effect (of body precession) is observed only during considerable times $(t \sim 1/\varepsilon)$.

6. The goemetric interpretation of motion. In the considered here cases it is possible to give a geometric interpretation of the motion of the body, which is similar to that given by Delauney /5/ to the problem of S. V. Kovalevskaia.

The first case. First of all, it should be noted the projection Q of the body instantaneous angular velocity on the vertical is

$$Q = \frac{p}{\varepsilon}\gamma + \frac{q}{\varepsilon}\gamma' + r\gamma'' = \frac{u}{\varepsilon} + r\gamma''$$

If at the initial instant $u \sim 1$, then $u = \text{const}$ (accurate within $\sim\varepsilon$). Thus the tip of the instantaneous angular velocity vector moves in a horizontal plane which oscillates along the vertical at amplitudes considerably smaller than the mean distance from that plane to the body fixed point. The latter means that in this approximation that plane can be taken as stationary.

Moreover, by virtue of the integral (with an accuracy to $\sim\varepsilon$).

$$\omega^2 = p^2 + q^2 = \text{const} \tag{6.1}$$

the tip of the instantaneous angular velocity vector moves on a cylinder rigidly attached to the body.

But the virtue of the integral

$$r^2 - \frac{2\xi}{\alpha\omega^2} up = \text{const} \tag{6.2}$$

the tip of that vector moves on a parabolic cylinder. Thus the directrix of the moving axoid is curve Γ of the intersection of the parabolic and circular cylinders (6.2) and (6.1), respectively.

In this approximation the body motion is represented by the rolling of curve Γ (rigidly attached to it) on the stationary horizontal plane.

Owing to the "narrowness" of the parabolic cylinder in comparison with the circular one, this interpretation reduces to the following. The body rotates almost uniformly about the vertical ($Q = \text{const}$), performing almost harmonic nutations (see the two last of Eqs.(4.6)) and rotates about the axis of symmetry similarly to a pendulum (see Eqs.(4.6) for $r, p,$ and q). And this constitutes the deviation from the motion by inertia.

The second case. In this case we formulate the geometric interpretation not directly to the investigated motion but to the motion of a cylinder H rotating in the body about their common axis of symmetry at the angular velocity $-k/\varepsilon$ (see (1.7) and (5.1)). When in this case the motion is by inertia, the absolute angular velocity (Ω_H) of the H cylinder is constant and proportional to the kinematic moment vector with the coefficient of proportionality equal $1/A$.

Using (5.6), (5.8), (5.9), and (5.1) we determine Ω_H taking into account the force of gravity. The projection of Ω_H on the axes of the coordinate system rotating together with the cylinder (the third axis coincides with the axis of symmetry) are (with the stated accuracy and a suitable selection of the initial position of equatorial axes)

$$\frac{I_*}{\varepsilon h}, \quad \frac{I}{\varepsilon h}, \quad R = \frac{(1-\alpha)n\omega}{\varepsilon} + \frac{2-\alpha}{\alpha n^2}s + \frac{(1-\alpha)^2}{\alpha}s_0, \quad h = \text{const}$$

Thus the interpretation of the motion of cylinder H is similar to that given in the first case.

By virtue of the integral $\varkappa = \text{const}$ the tip of the cylinder instantaneous angular velocity vector moves in a horizontal plane which in the considered approximation is stationary.

By virtue of integrals

$$I^2 + I_*^2 = \omega^2 h^2 = \text{const} \tag{6.3}$$

$$\frac{(2-\alpha)^2}{\alpha^2}\frac{\xi}{n^3\omega^2}I_* + \left(R - \frac{(1-\alpha)n\omega}{\varepsilon} - \frac{(1-\alpha)^2}{\alpha}s_0\right)^2 = \text{const} \tag{6.4}$$

the intersection line of the circular and parabolic cylinders (6.3) and (6.4), respectively, is the directrix of the moving axoid.

As in the first case, the effect of the force of gravity manifests itself by the pendulum-like motion of the body about its axis of symmetry.

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REFERENCES

1. ARNOL'D V.I., The proof of Kolmogorov's theorem on the conservation of conditionally-periodic motions at small variations of Hamilton's function. Uspekhi Matem. Nauk, Vol. 18, No.5, 1963.

2. KOZLOV V.V., New periodic solutions for the problem of motion of a heavy solid body around a fixed point. PMM, Vol.39, No.3, 1975.

3. GOLUBEV V.V., Lectures on the Integration of Equations of Motion of a Heavy Rigid Body about a fixed point. Moscow, Gostekhizdat, 1953.

4. POVZNER A., Linear methods in problems of nonlinear differential equations with a small parameter. Internat. J. Nonlinear Mech., Vol.9, No.4, pp.279-323, 1974.

5. DELAUNAY N.B., On the problem of geometric interpretation of integrals of motion of a rigid body about a fixed point, given by S. V. Kovalevskaia. Matem. Sb., 16, No.2, 1891.

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